Continuity

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Intuitively, **a continuous function** is often described as one whose graph can be drawn without lifting pencil from paper.

Before stating the precise definition of continuity, we illustrate some examples of graphs of functions that are not continuous (or discontinuous) at a number x_0.

Figures.

Definition

A function f is said to be **continuous** at a number x_0 if

- $f(x_0)$ is defined,
- $\lim_{x \to x_0} f(x)$ exists,
- $\lim_{x \to x_0} f(x) = f(x_0)$.

Example

The function $f(x) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$, $x \neq 1$, is discontinuous at 1 since $f(1)$ is not defined. However, f is continuous at any number $x \neq 1$.

Definition

A function f is said to be **continuous on an open interval (a,b)** if it is continuous at every number in the interval.

Definition

A function f is **continuous on a closed interval [a,b]** if it is continuous on (a,b), and, in addition,

$$\lim_{x \to a^+} f(x) = f(x_0) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

Extensions of these concepts to intervals such as (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$, $[a, b)$, $(a, b]$, $(-\infty, b]$ and $[a, \infty)$ are made in the expected manner.

Example

The function $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on the open interval $(-1, 1)$ but it is not continuous on the closed interval $[-1, 1]$ since neither $f(-1)$ nor $f(1)$ are defined.

Example

The function $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$. Let us observe that $\lim_{x \to -1^+} f(x) = f(-1) = 0$ and $\lim_{x \to 1^-} f(x) = f(1) = 0$.

Example

The function $f(x) = \sqrt{x-1}$ is continuous on $[1, \infty)$ since

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \sqrt{x-1} = \sqrt{x_0 - 1} = f(x_0) \quad \text{for} \quad x_0 > 1$$

and $\lim_{x \to 1^+} \sqrt{x-1} = f(1) = 0$.

Remark

A review of the graphs of trigonometric functions indicates that the sine and cosine functions are continuous on $(-\infty, \infty)$. The tangent and secant functions are discontinuous at $x = (2k+1)\frac{\pi}{2}$, $k = 0, \pm 1, \pm 2, \ldots$. The cotangent and cosecant functions are discontinuous at $x = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Theorem

If f and g are functions continuous at a number x_0, then cf, c a constant, $f+g$, fg, and f/g, $g(x_0) \neq 0$, are also continuous at x_0.

Corollary

A polynomial function is continuous on $(-\infty, \infty)$.

Corollary

A rational function $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions, is continuous at all numbers x for which $Q(x) \neq 0$.

Theorem

If $\lim_{x\to x_0} g(x) = L$ *and* f *is continuous at* L*, then*

$$\lim_{x\to x_0} f(g(x)) = f(\lim_{x\to x_0} g(x)) = f(L).$$

So, the composite function $f \circ g$ of two continuous functions f and g in continuous.

Theorem (Intermediate Value Theorem)

If f denotes a function continuous on a closed interval $[a, b]$ for which $f(a) \neq f(b)$, and if N is any number between $f(a)$ and $f(b)$, then there exists at least one number c between a and b such that $f(c) = N$.

So, the Intermediate Value Theorem states that a function f continuous on a closed interval $[a, b]$ takes on all values between $f(a)$ and $f(b)$. Put another way, f does not "skip" any values.

Corollary

If f satisfies the hypotheses of the above Theorem and $f(a)$ and $f(b)$ have opposite algebraic signs, then there exists some value of x between a and b for which $f(x) = 0$.

Remark

We often give a discontinuity of a function a special name.

- *If* $x = x_0$ *is a vertical asymptote for the graph of* $y = f(x)$*, then* f *is said to have an* **infinite discontinuity** *at* x_0*.*
- *If* $\lim_{x \to x_0^-} f(x) = L_1$ *and* $\lim_{x \to x_0^+} f(x) = L_2$ *and* $L_1 \neq L_2$*, then* f *is said to have a* **finite discontinuity** *or a* **jump discontinuity** *at* x_0
- *If* $\lim_{x \to x_0} f(x)$ *exists but* f *is either not defined at* x_0 *or* $f(x_0) \neq \lim_{x \to x_0} f(x)$*, then* f *is said to have a* **removable discontinuity** *at* x_0*.*

Example

The function $f(x) = \frac{x^2-1}{x-1} = x+1$, $x \neq 1$ is not defined at 1 but $\lim_{x\to 1} f(x) = 2$. By defining $f(1) = 2$, the new function

$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ is continuous at every number.