Derivative - part 1

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Definition

Let $y = f(x)$ be a continuous function. At a point $(x_0, f(x_0))$ the tangent line to the graph is the line that passes through the point with slope

$$m_{tan} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

whenever the limit exists.

The slope of the tangent line at $(x_0, f(x_0))$ is also called the slope of the curve at the point. The tangent line to a graph at a point can be vertical in which case its slope is undefined.

The graph of a function f will not have a tangent line at a point whenever:

- f is discontinuous at $x = x_0$, or
- the graph of f has a corner at $(x_0, f(x_0))$, or
- the graph has a sharp peak.

Example

Show that the graph of the function $f(x) = |x|$ does not have a tangent at (0,0).

Definition

The derivative of a function $y = f(x)$ with respect to x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

whenever the limit exists.

Remark

The following is a list of some of the symbols used throughout mathematical literature to denote the derivative of a function:
$f'(x)$, $\frac{dy}{dx}$, y', $\dot{y}$, Dy, $D_x y$.

Example

- $(C)' = 0$, where C is a constant,
- $(x^2)' = 2x$, $(\sin(x))' = \cos(x)$.

Definition

At a specified number x_0, the value of the derivative is denoted by $f'(x_0)$.

Definition

The process of finding a derivative is called differentiation. If $f'(x_0)$ exists, the function f is said to be differentiable at x_0.

Remark

If $y = f(x)$ is continuous at $x = x_0$ and $f'(x_0) = 0$, then the tangent line at $(x_0, f(x_0))$ is horizontal.

Definition

The domain of f' is the set of numbers x for which the limit exists. A derivative fails to exist at a number x_0 for the same reasons a tangent line to its graph fails to exist:

- the function is discontinuous at $x = x_0$, or
- the graph could have a sharp peak or corner at $(x_0, f(x_0))$, or
- the tangent line to the graph is vertical.

The domain of f' is necessarily a subset of the domain of f.

Example

1. $f(x) = \frac{1}{2x-1}$ is discontinuous at 0.5, so f is not differentiable at this value.
2. The graph of $f(x) = |x|$ possesses a sharp corner at the origin. Earlier we have shown that $f'(0)$ does not exist so f is not differentiable there.
3. The graph of $f(x) = x^{1/3}$ possesses a vertical tangent at the origin. Hence, the function f is not differentiable at 0.

Definition

A function f is said to be differentiable

- on an open interval (a,b) when f' exists for every number in the interval,
- on a closed interval [a,b] when f is differentiable on (a,b) and

$$f'_+(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x) - f(a)}{\Delta x}, \quad f'_-(b) = \lim_{\Delta x \to 0^-} \frac{f(b+\Delta x) - f(b)}{\Delta x}$$

  both exist.

Those limits are called right-hand and left-hand derivatives, respectively.

Theorem

If f is differentiable at a number x_0, then f is continuous at x_0.

Remark

1. *Continuity does not imply differentiability (e.g. $f(x) = |x|$).*
2. *If f is differentiable at x_0, then an equation of the tangent line at $(x_0, f(x_0))$ is:*

$$y - f(x_0) = f'(x_0)(x - x_0).$$

3. *A normal line to a graph at a point P is one that is perpendicular to the tangent line at P.*

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0).$$

The list of derivatives for elementary functions

- $(C)' = 0$, C – constant;
- $(x^{\alpha})' = \alpha x^{\alpha-1}$, $\alpha \in \mathbb{R}$;
- $(a^x)' = a^x ln(x)$;
- $(e^x)' = e^x$;
- $(log_a(x))' = \frac{1}{xln(x)}$;
- $(ln(x))' = \frac{1}{x}$;
- $(\sin(x))' = \cos(x)$;
- $(\cos(x))' = -\sin(x)$;
- $(\tan(x))' = \frac{1}{\cos^2(x)}$;
- $(\cot(x))' = \frac{-1}{\sin^2(x)}$;

- $(asin(x))' = \frac{1}{\sqrt{1-x^2}}$;
- $(acos(x))' = \frac{-1}{\sqrt{1-x^2}}$;
- $(atan(x))' = \frac{1}{1+x^2}$;
- $(acot(x))' = \frac{-1}{1+x^2}$;

Theorem

If f and g are differentiable functions, then cf, c a constant, $f+g$, fg, and f/g (where $g(x) \neq 0$) are also differentiable functions.

Rules of differentiations

- Power Rule: $(x^{\alpha}) = \alpha x^{\alpha-1}$ (bring down exponent as a multiple and decrease exponent by 1)
- Constant Multiple of a Function: $(c \cdot f(x))' = c \cdot f'(x)$
- Sum Rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ (The Sum Rule extends to any finite sum of functions. Thus, any polynomial function is differentiable.)

- Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ (The Product Rule is usually memorized in words: the derivative of the first function times the second plus the first function times the derivative of the second)
- Quotient Rule: $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$ (the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator all divided by the denominator squared)
- Chain Rule for Compound Functions: If $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then

$$(f(g(x)))' = f'(g(x))g'(x).$$

Theorem

In general, we say that the limits:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}, \quad \lim_{x \to \infty} \frac{f(x)}{g(x)}, \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)}$$

are indeterminate forms if, as x approaches x_0, ∞, or $-\infty$, $(f(x) \to 0$ and $g(x) \to 0)$ or $(|f(x)| \to \infty$ and $|g(x)| \to \infty)$. Symbolically, we denote an indeterminate form as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theorem (de l'Hospital Rule)

Suppose $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ *is an indeterminate form and the limit* $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ *exists. Then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

De l'Hospital Rule can be applied to appropriate one-sided limits.

The Forms $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0 and 1^∞

In these cases we use the following transformations:

$$f_0\cdot g_\infty=\frac{f_0}{\frac{1}{g_\infty}}=\frac{f_0}{h_0};\quad f_0\cdot g_\infty=\frac{g_\infty}{\frac{1}{f_0}}=\frac{g_\infty}{k_\infty};$$

$$f_\infty-g_\infty=\frac{1}{\frac{1}{f_\infty}}-\frac{1}{\frac{1}{g_\infty}}=\frac{\frac{1}{g_\infty}-\frac{1}{f_\infty}}{\frac{1}{f_\infty}\cdot\frac{1}{g_\infty}}=\frac{k_0-h_0}{h_0\cdot k_0}.$$

In cases 0^0, ∞^0 and 1^∞ we use the relation:

$$(f(x))^{g(x)} = e^{g(x) lnf(x)}.$$

So,

$$\lim_{x \to x_0} (f(x))^{g(x)} = e^{\lim\limits_{x \to x_0} g(x) lnf(x)}.$$

Example

1. $\lim_{x\to 0} \frac{e^x - e^{-x} + 3x}{x}$;
2. $\lim_{x\to 0^+} \frac{ln(\sin(x))}{ln(\sin(2x))}$;
3. $\lim_{x\to 0^-} (\frac{1}{x} - \frac{1}{e^x - 1})$;
4. $\lim_{x\to 1^+} (x-1)ln(x-1)$;
5. $\lim_{x\to 1^+} (x-1)^{(x-1)}$;
6. $\lim_{x\to 2^+} (ln(\frac{1}{x-2}))^{x-2}$;
7. $\lim_{x\to \frac{\pi}{4}^+} (\tan(x))^{\tan(2x)}$.