Derivative - part 2

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EXTREMA OF FUNCTIONS

Definition

Suppose a function f is defined on an interval I. The maximum and minimum values of f on I (if there are any) are said to be extrema of the function.

Definition

- A number $f(c_1)$ is an absolute maximum of a function f if $f(x) \leq f(c_1)$ for every x in the domain of f.
- A number $f(c_1)$ is an absolute minimum of a function f if $f(x) \geq f(c_1)$ for every x in the domain of f.

Absolute extrema are also called global extrema. Figures.

Theorem (Extreme Value Theorem)

A function f continuous on a closed interval $[a, b]$ always has an absolute maximum and an absolute minimum on the interval.

Definition

When an absolute extremum of a function occurs at an endpoint of an interval I, we say it is an endpoint extremum. When I is not a closed interval (e.g. such as $[a, b)$, $(-\infty, b]$ or $[a, \infty)$), then even when f is continuous there is no guarantee that an absolute extremum exists.

Figures.

Definition (Relative extrema (or local extrema))

- A number $f(c_1)$ is a relative maximum of a function f if $f(x) \leq f(c_1)$ for every x in some open interval that contains c_1.
- A number $f(c_1)$ is a relative minimum of a function f if $f(x) \geq f(c_1)$ for every x in some open interval that contains c_1.

Figure.

Remark

An examination of the figure suggests that if c is a value at which a function f has a relative extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition

A critical value of a function f is a number c in its domain for which $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem

If a function f has a relative extremum at a number c, then c is a critical value.

Theorem

If f is continuous on a closed interval $[a, b]$, then an absolute extremum occurs either at an endpoint of the interval or at a critical value in the open interval (a, b).

Finding the absolute extrema

To find an absolute extremum of a function f continuous on $[a, b]$

- Evaluate f at a and b.
- Find all critical values c_1, c_2, ..., c_n in (a, b).
- Evaluate f at all critical values.
- The largest and the smallest values in the list: $f(a)$, $f(b)$, $f(c_1)$, $f(c_2)$, ..., $f(c_n)$ are the absolute maximum and the absolute minimum, respectively, of f on the interval $[a, b]$.

Example

1. Find the critical values of the given functions: $f(x) = (x-1)^2\sqrt[3]{x+2}$ and $f(x) = \frac{x+4}{\sqrt[3]{x+1}}$.
2. Find the absolute extrema of the given function on the indicated interval: $f(x) = x^3 - 3x^2 + 3x - 1$ on $[-4, 3]$ and $f(x) = x^4(x-1)^2$ on $[-1, 2]$.

Theorem (Rolle's Theorem)

Let f be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b) = 0$. Then there exists a number c in (a,b) such that $f'(c) = 0$.

Theorem (Lagrange's Theorem - Mean Value Theorem for Derivatives)

Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Figure.
Geometrically, the Mean Value Theorem asserts that the slope of the tangent line at $(c, f(c))$ is the same as the slope of the secant through $(a, f(a))$ and $(b, f(b))$.

Remark

Let us remember that:

- *a function f is said to be increasing on I if $f(x_1) < f(x_2)$ when x_1 and x_2 are numbers in I that satisfy $x_1 < x_2$,*
- *a function f is said to be decreasing on I if $f(x_1) > f(x_2)$ when x_1 and x_2 are numbers in I that satisfy $x_1 < x_2$.*

Theorem

If $f'(x) = 0$ for all x in an interval $[a, b]$, then $f(x)$ is a constant on the interval.

Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b).

- *If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).*
- *If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).*

Example

Determine the intervals on which the given function f is increasing and the intervals on which f is decreasing:

$$f(x) = x\sqrt{8 - x^2} \qquad f(x) = xe^{-\frac{1}{2}x^2}.$$

By differentiating the first derivative $f'(x)$, we obtain another function called the second derivative which is denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$, we define the second derivative with respect to x as $\frac{d}{dx}(\frac{df}{dx})$. Assuming all derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. So, the third derivative is the derivative of the second derivative. The fourth derivative is the derivative of the third derivative, and so on. In general, if n is a positive integer, then the *nth* derivative of the function $y = f(x)$ is defined by

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right).$$

Figures below illustrate geometric shapes that are concave upward and concave downward, respectively. Often a shape that is concave upward is said to "hold water", whereas a shape that is concave downward is said to "spill water".

Theorem

Let f be a function for which f'' exists on (a, b).

- *If $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is concave upward on (a, b).*
- *If $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is concave downward on (a, b).*

Definition

Let f be continuous at c. A point $(c, f(c))$ is a point of inflection if there exists an open interval (a, b) that contains c such that the graph of f is either

- concave upward on (a, c) and concave downward on (c, b) OR
- concave downward on (a, c) and concave upward on (c, b).

Corollary

A point of inflection $(c, f(c))$ occurs at a number c for which $f''(x) = 0$ or $f''(c)$ does not exist.

Example

Check the concavity of the functions: $f(x) = ln(x) + 3x - 1$ and $f(x) = xe^x$.

Theorem (Second Derivative Test for Relative Extrema)

Let f be a function for which f'' exists on an interval (a, b) that contains the critical number c.

- *If $f''(c) > 0$, then $f(c)$ is a relative minimum.*
- *If $f''(c) < 0$, then $f(c)$ is a relative maximum.*

Homework

Check the behaviour of the given functions:

1. $f(x) = x^3(x+1)^2$
2. $f(x) = \frac{x}{x^2+2}$
3. $f(x) = x\sqrt{x-6}$