Indefinite integrals

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Indefinite integrals

Until now we were concerned with the basic problem: "Given a function f find its derivative f'". But now we have an equally important problem: "Given a function f find a function whose derivative is the same as f".

Definition

A function F is said to be an antiderivative of a function f if $F'(x) = f(x)$ on some interval.

Example

An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$ since $F'(x) = x^2$ (but also $F(x) = \frac{1}{3}x^3 + 10$, $F(x) = \frac{1}{3}x^3 - \pi$ and so on).

Theorem

If $G'(x) = F'(x)$ for all x in some interval $[a, b]$, then $G(x) = F(x) + C$, where C is a constant, for all x in the interval.

Definition

For convenience let's introduce a notation for an antiderivative of a function. If $F'(x) = f(x)$, we represent the most general antiderivative of f by

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is called an integral sign, and the notation $\int f(x)dx$ is called the indefinite integral of $f(x)$ with respect to x. The function $f(x)$ is called the integrand. The process of finding an antiderivative is called antidifferentiation or integration. The number C is called a constant of integration.

Integrals of Elementary Functions

- $\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C$, where $\alpha \neq -1$
- $\int \frac{1}{x} dx = \ln|x| + C$,
- $\int a^x dx = \frac{a^x}{\ln(a)} + C \Rightarrow \int e^x dx = e^x + C$,
- $\int \sin(x) dx = -\cos(x) + C$,
- $\int \cos(x) dx = \sin(x) + C$,
- $\int \frac{1}{\cos^2(x)} dx = \tan(x) + C$,
- $\int \frac{1}{\sin^2(x)} dx = \cot(x) + C$,
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$,
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$.

Theorem (Linearity of Indefinite Integrals)

If $F'(x) = f(x)$ and $G'(x) = g(x)$, and a, b are constants, then:

$$\int a \cdot f(x) \pm b \cdot g(x) dx = a \cdot \int f(x) dx \pm b \cdot \int g(x) dx =$$

$$= a \cdot F(x) \pm b \cdot G(x) + C.$$

Example

- $\int (x^3 + \frac{2}{x^2} - 5\sqrt{x} + \frac{7}{\sqrt[5]{x^3}}) dx$,
- $\int \frac{x^2-5}{x^2+1} dx$, $\int \frac{1}{\sin^2(x)\cos^2(x)} dx$,
- $\int \frac{\cos(2x)}{\cos(x)-\sin(x)} dx$, $\int \frac{1+\cos^2(x)}{1+\cos(2x)} dx$.

Theorem (Integration by substitution)

If F is an antiderivative of f, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Example

1. $\int \frac{1}{ax+b}dx = \left| \begin{array}{l} t = ax+b \\ dt = a \cdot dx \end{array} \right| = \int \frac{dt}{a \cdot t} = \frac{1}{a}\ln|t| + C =$
   $\frac{1}{a}\ln|ax+b| + C,$
2. $\int \frac{f'(x)}{f(x)}dx = \left| \begin{array}{l} t = f(x) \\ dt = f'(x) \cdot dx \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C.$

Example

Evaluate the integrals: $\int \frac{1}{2x+5}dx$, $\int \frac{2x+5}{x^2+5x+12}dx$, $\int \frac{1}{(7x+13)^{29}}dx$, $\int \frac{1}{\sqrt{5x+7}}dx$, $\int x\sqrt{x^2+5}dx$, $\int \frac{x^3}{\sqrt[5]{x^4+3}}dx$, $\int \tan(x)dx$, $\int \cot(x)dx$, $\int \sin^5(x)\cos(x)dx$, $\int \sin(7x)dx$, $\int \frac{1}{x^2+25}dx$, $\int \frac{e^x}{1+e^{2x}}dx$, $\int \frac{e^{2x}}{e^x-1}dx$, $\int \frac{\cos(x)}{\sqrt{4-9\sin^2(x)}}dx$.

Theorem (Integration by Parts)

If the functions f and g are differentiable in some interval (a, b) *then*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Example

Evaluate the integrals: $\int x \cdot \cos(x)dx$, $\int \ln(x)dx$, $\int x \cdot \ln(x)dx$, $\int \arctan(x)dx$, $\int e^x \cdot \sin(x)dx$.

Integration of Trigonometric Functions

Example

Evaluate the integrals: $\int \sin(x)dx$, $\int \cos(x)dx$, $\int \frac{1}{\cos^2(x)}dx$, $\int \frac{1}{\sin^2(x)}dx$, $\int \sin(x)\cos(x)dx$, $\int \sin^2(x)dx$, $\int \cos^2(x)dx$, $\int \sin^2(x)\cos^2(x)dx$, $\int \sin^3(x)dx$, $\int \sin^3(x)\cos^2(x)dx$, $\int \sin^2(x)\cos^5(x)dx$, $\int \sin^3(x)\cos^3(x)dx$.

The recurrence formulas:

$$\int \sin^n(x)dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$$

and

$$\int \cos^n(x)dx = \frac{1}{n}\sin(x)\cos^{n-1}(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx.$$

Example

Evaluate the integrals: $\int \sin^4(x)dx$ and $\int \cos^6(x)dx$.

The reduction formulas:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta)),$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)),$$

$$\sin(\alpha)\sin(\beta) = -\frac{1}{2}(\cos(\alpha+\beta)-\cos(\alpha-\beta)).$$

Example

Evaluate the integrals: $\int \sin(3x)\cos(2x)dx$, $\int \cos(5x)\cos(7x)dx$, and $\int \sin(5x)\sin(2x)dx$.

Universal Substitution:

$$t = \tan(\tfrac{x}{2}) \qquad \sin(x) = \frac{2t}{1+t^2}$$
$$dx = \frac{2}{1+t^2}dt \quad \cos(x) = \frac{1-t^2}{1+t^2}$$

Example

Evaluate the integrals: $\int \frac{dx}{2+\cos(x)}$ and $\int \frac{dx}{1+\sin(x)+\cos(x)}$.

Integration of Rational Functions

Example

1. $\int \frac{1}{x} dx$, $\int \frac{1}{ax+b} dx$, $\int \frac{1}{(ax+b)^k}$
2. $\int \frac{1}{ax^2+bx+c} dx$
   - $\Delta = 0$: $\int \frac{dx}{9x^2-12x+4}$
   - $\Delta < 0$: $\int \frac{dx}{2x^2-12x+27}$
   - $\Delta > 0$: $\int \frac{dx}{3x^2-5x-2}$
3. $\int \frac{ex+f}{ax^2+bx+c} dx$
   - $\Delta = 0$: $\int \frac{8x-5}{9x^2-6x+1} dx$
   - $\Delta < 0$: $\int \frac{3x+1}{x^2+4x+5} dx$
   - $\Delta > 0$: $\int \frac{11x-1}{3x^2-5x-2} dx$